

ARITHMETICITY OF CERTAIN SYMPLECTIC HYPERGEOMETRIC GROUPS

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ABSTRACT. We give a sufficient condition on a pair of (primitive) polynomials that the associated hypergeometric group (monodromy group of the corresponding hypergeometric differential equation) is an arithmetic subgroup of the integral symplectic group.

1. INTRODUCTION

We will first state our main theorem in simple terms and then explain the context in which it arises.

Let $f, g \in \mathbb{Z}[X]$ be monic polynomials of degree n which are *reciprocal* i.e. $X^n f(\frac{1}{X}) = f(X)$, $X^n g(\frac{1}{X}) = g(X)$. Write (in the following, note that $A_0 = B_0 = 1$)

$$f(X) = X^n + A_{n-1}X^{n-1} + \cdots + A_1X + A_0,$$

$$g(X) = X^n + B_{n-1}X^{n-1} + \cdots + B_1X + B_0.$$

Form the companion matrices A and B of f, g respectively. Then

$$A = \begin{pmatrix} 0 & \cdots & 0 & -A_0 \\ 1 & \cdots & 0 & -A_1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & -A_{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & -B_0 \\ 1 & \cdots & 0 & -B_1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & -B_{n-1} \end{pmatrix}.$$

It is clear from our assumptions that the group Γ generated by A, B lies in $\mathrm{SL}_n(\mathbb{Z})$.

We will assume further, that f, g do not have a common root in \mathbb{C} . Assume also that f, g is a *primitive pair*. That is, there does not exist an integer $k \geq 2$ such that $f(X) = f_1(X^k)$, $g(X) = g_1(X^k)$ for some polynomials $f_1, g_1 \in \mathbb{Z}[X]$. It is then known by a criterion of Beukers and Heckman ([BH]) that Γ preserves a non-degenerate integral symplectic form Ω on \mathbb{Z}^n and that $\Gamma \subset \mathrm{Sp}_\Omega$ is Zariski dense.

The following question has been considered by many people ([S]) ; characterise the polynomials f, g for which the group Γ is an arithmetic

group (i.e. of finite index in $\mathrm{Sp}_\Omega(\mathbb{Z})$)?

In this paper we give a *sufficient* condition for arithmeticity of Γ . To describe the condition, set $h = f - g$ the difference polynomial. This is a polynomial with integer coefficients:

$$h(X) = cX^d + c_{d-1}X^{d-1} + \cdots + c_rX^r,$$

for some $d \leq n - 1$ and some r with $1 \leq r \leq d$, with the leading coefficient of h denoted c .

Theorem 1.1. *Suppose $n \geq 4$. In addition to the foregoing assumptions on the polynomials f, g , assume that the leading coefficient c of h satisfies*

$$|c| \leq 2.$$

Then the group Γ is an arithmetic group.

In particular, if the difference $h = f - g$ is monic, then the group Γ is arithmetic.

Theorem 1.1 holds even when $n = 2$ but this is easy (cf. Lemma 3.3) and therefore we do not describe the proof. For $n \geq 4$, Theorem 1.1 is proved by showing that the “reflection subgroup” ([BH]) Γ_r generated by the conjugates $\{A^{-k}CA^k; \ k \in \mathbb{Z}\}$ of the element $C = A^{-1}B$ is arithmetic; the inclusion $\Gamma_r \subset \Gamma \subset \mathrm{Sp}_\Omega(\mathbb{Z})$ then shows that Γ is arithmetic. The element C is a *complex reflection* i.e. it is identity on a codimension one subspace of \mathbb{Q}^n (it is also called a *transvection*, in the symplectic case). We will deduce the arithmeticity of the reflection subgroup Γ_r from the following result on subgroups of $\mathrm{Sp}(\mathbb{Z})$ generated by certain transvections.

Suppose Ω is a non-degenerate symplectic form on \mathbb{Q}^n which is integral on the standard lattice \mathbb{Z}^n . Suppose that $\Gamma \subset \mathrm{Sp}_\Omega(\mathbb{Z})$ is a Zariski dense subgroup which contains three transvections C_1, C_2, C_3 such that if we write $\mathbb{Z}e_i = (C_i - 1)(\mathbb{Z}^n)$, then $\Omega(e_i, e_j) \neq 0$ for some i, j . Assume that $W = \sum_{i=1}^3 \mathbb{Q}e_i$ is three dimensional, and that the group generated by the restrictions of the C_i to W ($i = 1, 2, 3$) contains a non-trivial (i.e. non-identity) element of the unipotent radical of the symplectic group of W .

Theorem 1.2. *Under the preceding assumptions, the group Γ is of finite index in $\mathrm{Sp}_\Omega(\mathbb{Z})$.*

In Section 3, Theorem 1.1 is deduced from Theorem 1.2 by checking that three generic conjugates of the transvection $C = A^{-1}B$ satisfy the

conditions of Theorem 1.2, under the assumption that the difference polynomial $f - g$ has leading coefficient at most two in absolute value. In the course of the proof we have to deal with an associated subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that we get from the group Γ , which is generated by the 2×2 matrices

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

The assumption that the leading coefficient c is ≤ 2 in absolute value, implies that this subgroup is of finite index in $\mathrm{SL}(2, \mathbb{Z})$. We use this fact in the proof of arithmeticity. This is the reason that we are unable to extend the proof to other c 's.

In Section 2 Theorem 1.2 will be proved by showing that with respect to a suitable basis, the group Γ intersects the highest and the second highest root groups of the symplectic group Sp_Ω non-trivially (and using Theorem (3.5) of [Ve]).

In the last section, we give tables for pairs (f, g) of degree 4 monic polynomials with integer coefficients, for which the group Γ is Zariski dense in Sp_4 , and which have roots of unity as roots. We list those pairs for which the criterion of Theorem 1.1 applies (and also those for which it does not).

In [YYCE], pp 171-172, fourteen pairs of polynomials f, g which are coprime, primitive and have roots of unity as roots, are considered. In Example 13 of [YYCE], $g(X) = (X - 1)^4$ and $f(X) = (X^2 - X + 1)^2$ so that the roots of f are of the form $e^{2\pi i \alpha_j}$ with $\alpha_1 = \alpha_2 = \frac{1}{6}, \alpha_3 = \alpha_4 = \frac{5}{6}$. This pair satisfies the conditions of Theorem 1.1 (see the first example of Table 4.1) and hence we have, from Theorem 1.1 the following

Corollary 1.3. *For the polynomials $f(X) = (X^2 - X + 1)^2$ and $g(X) = (X - 1)^2$, the group $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ is arithmetic.*

Note that no other example in [YYCE] satisfies the hypothesis of the Theorem 1.1.

Remarks 1.4. [1] The group Γ of Theorem 1.1 is a *hypergeometric group* in the sense of Beukers-Heckman ([BH]) i.e. it is the monodromy group of a hypergeometric equation of type ${}_nF_{n-1}$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with regular singularities at $0, 1, \infty$ and regular elsewhere. We do not know a good criterion for the arithmeticity of the hypergeometric group Γ , when the leading coefficient c is more than two in absolute value.

[2] A recent result by Brav and Thomas ([BT]) says that when $n = 4$ and $f(X) = \frac{X^5-1}{X-1}$ and $g(X) = (X-1)^4$ (in this case, all the coefficients of h are ± 5), the group Γ is *thin* in the sense of [FMS]). That is, Γ is of infinite index in the integral symplectic group.

[3] An analogous question can be asked when f, g are as before, but $f(0) = 1$ and $g(0) = -1$. Then, by [BH], the group Γ preserves a non-degenerate *quadratic form* q with integral coefficients and the Zariski closure of Γ is $O(n)$. Then the question would be whether the group $\Gamma \subset O(q)(\mathbb{Z})$ is of finite index in the integral orthogonal group $O(q)(\mathbb{Z})$. If the index of the quadratic form q over \mathbb{R} is $(n-1, 1)$, then [FMS] give an infinite family of examples when Γ is of infinite index (i.e. is *thin*), producing perhaps the first examples of thin monodromy groups with absolutely simple Zariski closure (namely, $SO(n)$). If the index is (p, q) with $p, q > 1$, then we do not know a criterion analogous to Theorem 1.1.

[4] The hypergeometric monodromy group Γ is relevant to algebraic geometry when f, g are products of cyclotomic polynomials (i.e. when their roots are roots of unity). This is equivalent to saying that the “local monodromy” A and B are quasi-unipotent elements (i.e. some power of A and B are unipotent). Theorem 1.1 says for example that if $f(X) = \frac{X^5-1}{X-1}$ and $g(X) = (X^2+1)^2$, then Γ is a subgroup of finite index in $Sp_4(\mathbb{Z})$. Similar examples can be constructed for any even integer n .

For example, let $m \geq 2$ be an integer and let

$$f(X) = \frac{X^{2m+1} - 1}{X - 1}, \quad g(X) = (X^m - 1)^2.$$

Then f, g are coprime and form a primitive pair. Hence the associated Γ is an arithmetic group.

In [Ve2], we prove, by a different method, that the hypergeometric group Γ associated to the polynomials (n is even)

$$f(X) = \frac{X^{n+1} - 1}{X - 1}, \quad g(X) = \frac{(X - 1)(X^n - 1)}{X + 1}$$

is an arithmetic subgroup of $Sp_n(\mathbb{Z})$. This also follows from Theorem 1.1 since the leading coefficient of the difference $f - g$ is ± 2 .

Our computations show that when $n = 4$, then for most polynomials f, g (more than 50 examples) the hypotheses of Theorem 1.1 are satisfied and hence the monodromy is arithmetic.

However, in Theorem 1.1, we do not assume that the roots of f, g are roots of unity, since the proof holds without this assumption.

[5] Theorem 1.2 is somewhat analogous to the criterion of Jannsen ([Jan]). It can be checked that if the assumption (on “vanishing lattices” having vectors δ_1, δ_2 with dot-product 1) of Jannsen’s paper holds in our case, then the difference polynomial $h = f - g$ is monic. The proof of Theorem 1.2 shows that Jannsen’s Theorem holds true if we assume that $\delta_1 \cdot \delta_2 = 2$ for two vectors in the vanishing lattice in the sense of [Jan].

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2. PROOF OF THEOREM 1.2

2.1. Subgroups of a semi-direct product. Consider the natural action of the integral linear group $SL(2, \mathbb{Z})$ on \mathbb{Z}^2 . Form the semi-direct product $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ and suppose $\Delta \subset \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is a subgroup, whose projection to $SL(2, \mathbb{Z})$ is Zariski dense in $SL(2, \mathbb{C})$, and suppose that Δ contains a non-trivial element of \mathbb{Z}^2 .

Lemma 2.1. *The intersection of Δ with the integral unipotent group \mathbb{Z}^2 is of finite index in \mathbb{Z}^2 .*

Proof. Since \mathbb{Z}^2 is Abelian, The action of Δ on \mathbb{Z}^2 factors through its projection to $SL(2, \mathbb{Z})$. By assumption, Δ has Zariski dense image in $SL(2)$, and hence acts irreducibly on \mathbb{Q}^2 . Since $\Delta \cap \mathbb{Z}^2$ is non-zero, it follows that the normal subgroup generated by $\Delta \cap \mathbb{Z}^2$ in Δ contains two \mathbb{Q} -linearly independent elements in \mathbb{Z}^2 , i.e. $\Delta \cap \mathbb{Z}^2$ has finite index in \mathbb{Z}^2 . \square

We will apply this Lemma to the following situation. Let W be a three dimensional \mathbb{Q} -vector space with a *non-zero* symplectic form Ω on W . Since W is odd dimensional, Ω is degenerate and hence W has a null subspace, which must be one dimensional: $E = \mathbb{Q}e$ for some $e \in W \setminus \{0\}$.

Then the symplectic group Sp_W of Ω is not reductive, and is in fact a semi-direct product: $Sp_W(\mathbb{Q}) = \mathbb{Q}^2 \rtimes SL(2, \mathbb{Q})$, where $\mathbb{Q}^2 = Hom(W/E, E)$ is identified with the unipotent radical U of Sp_W by sending a linear form $\lambda \in Hom(W/E, E)$ to the (symplectic) unipotent linear transformation $x \mapsto x + \lambda(x)$, with $x \in W$.

Moreover, $SL(2, \mathbb{Q}) \simeq Sp_{W/E}$ is the symplectic group of the non-degenerate form defined by Ω on the quotient W/E .

Suppose that $W_{\mathbb{Z}}$ denotes the integral span of a basis of W , and suppose that $w_1, w_2, w_3 \in W_{\mathbb{Z}}$ are linearly independent over \mathbb{Q} . Denote by C'_i the transvection

$$x \mapsto x - \Omega(x, w_i)w_i \quad \forall x \in W.$$

Denote by Δ' the group generated by the transvections $C'_1, C'_2, C'_3 \subset Sp_W(\mathbb{Z})$.

Lemma 2.2. *If Δ' contains a non-trivial element of $U(\mathbb{Z})$, then Δ' contains a subgroup of finite index in $U(\mathbb{Z})$.*

Proof. We will apply Lemma 2.1. Since Ω is not the zero symplectic form on W , and w_1, w_2, w_3 is a basis of W , it follows that $\Omega(w_i, w_j) \neq 0$

for some $i \neq j$; we may assume, after a renumbering, that $\Omega(w_1, w_2) \neq 0$. Hence the span of w_1, w_2 does not intersect the null space $E = \mathbb{Q}e$, and e, w_1, w_2 form a basis of W . We assume, as we may, that $e \in W_{\mathbb{Z}}$. Replacing Δ' by a subgroup of finite index, we may assume that Δ' preserves the lattice in $W_{\mathbb{Q}}$, spanned by e, w_1, w_2 . Since the unipotent radical U consists only of (unipotent and hence) elements of infinite order, the assumptions of the lemma are not altered if we replace Δ' by a finite-index subgroup.

With respect to this basis e, w_1, w_2 , the group generated by C'_1, C'_2 fixes e_1 and takes w_1, w_2 into linear combination of w_1, w_2 . Write $\lambda = \Omega(w_1, w_2) \neq 0$. Then, the matrices of C'_1, C'_2 are

$$C'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}, \quad C'_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\lambda & 1 \end{pmatrix}.$$

It is therefore clear that the group generated by C'_1, C'_2 is Zariski dense in $\mathrm{SL}(2, \mathbb{C}) \simeq \mathrm{Sp}_W/U$.

Hence the assumptions of Lemma 2.1 are satisfied and so is the conclusion. \square

2.2. A Unipotent Subgroup of Sp_4 . Suppose that X is a four dimensional \mathbb{Q} -vector space with a non-degenerate symplectic form Ω_X . Suppose C_1, C_2, C_3 are three transvections corresponding to vectors w_1, w_2, w_3 which are linearly independent, and such that $\Omega_X(w_1, w_2) \neq 0$. If W is the span of the w_i and C'_i denotes the restriction of C_i to W , then we assume that the hypotheses of Lemma 2.2 are satisfied.

As before, let U be the unipotent radical of Sp_W , and $\mathbb{Q}e$ be the null space of Ω restricted to W . Denote by P_X the subgroup of the 4×4 symplectic group Sp_X , which preserves the partial flag $\mathbb{Q}e \subset W \subset X$. Then P_X is a parabolic subgroup of Sp_X . The subgroup of P_X which acts trivially on successive quotients of this flag is precisely its unipotent radical, denoted U_X . We have the restriction map $P_X \rightarrow \mathrm{Sp}_W$, which is easily seen to be surjective.

The group generated by C_1, C_2, C_3 lies in P_X : since W is the span of the images $(C_i - 1)$, and C_i are transvections, it follows that W is stable under the C_i 's.

It is also trivial to see that U_X is the preimage of U under the restriction map $P_X \rightarrow \text{Sp}_W$ and that the kernel of the surjective map $U_X \rightarrow U$ is the commutator subgroup $[U_X, U_X]$ (which is one dimensional).

If $e \in W$ generates the null space of W , the non-degeneracy of X shows the existence of $e^* \in X$ such that $\Omega(e, e^*) \neq 0$. Denote by $X_{\mathbb{Z}}$ the integral span of the vectors e, w_1, w_2, e^* . Thus $\text{Sp}_X(\mathbb{Z})$ is the space of symplectic transformations on X which preserves this integral span $X_{\mathbb{Z}}$. If we choose a different lattice in $X_{\mathbb{Q}}$, then the resultant integral symplectic group is commensurable with $\text{Sp}_X(\mathbb{Z})$.

With this notation, we have the following Lemma.

Lemma 2.3. *The group Δ generated by C_1, C_2, C_3 contains a finite-index subgroup of $U_X(\mathbb{Z})$.*

Proof. The group Δ maps onto the group Δ' of Lemma 2.2. By Lemma 2.2, Δ' contains a finite-index subgroup U'_1 of $U(\mathbb{Z})$. The preimage of U'_1 in P_X is generated by the kernel $[U_X, U_X]$ and a finite-index subgroup of $U_X(\mathbb{Z})$. Hence $\Delta \cap U_X$ maps onto a finite-index subgroup of $U'(\mathbb{Z})$. Therefore, $\Delta \cap U_X$ is of finite index in $U_X(\mathbb{Z})$. \square

Corollary 2.4. *In particular, Δ contains the subgroup of matrices of the form*

$$\begin{pmatrix} 1 & 0 & y_2 & z \\ 0 & 1 & 0 & \frac{\lambda_1}{\lambda_2} y_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with y_2, z in a subgroup of finite index in \mathbb{Z} .

In the above corollary $\lambda_1 = \Omega(e, e^*)$ and $\lambda_2 = \Omega(w_1, w_2)$.

2.3. Proof of Theorem 1.2. We will now prove Theorem 1.2. Let C_1, C_2, C_3 be three transvections satisfying the conditions of Theorem 1.2. They correspond to linearly independent vectors $w_i \in \mathbb{Z}^n$. Set W to be their \mathbb{Q} -span, and let, as before, e be a generator of the null space in W of the symplectic form Ω . The non-degeneracy of \mathbb{Z}^n as a symplectic space implies the existence of a vector $e^* \in \mathbb{Z}^n$ such that $\Omega(e, e^*) \neq 0$. Let X be the span of W and $\mathbb{Q}e^*$. We may write the orthogonal decomposition

$$\mathbb{Z}^n = X \oplus X^\perp.$$

Hence the “reflections” C_i act trivially on X^\perp . Write $\epsilon_1 = e$, $\epsilon_2 = w_1$, $\epsilon_2^* = w_2$, $\epsilon_1^* = e^*$. Now a “symplectic” basis $\epsilon_3, \dots, \epsilon_n; \epsilon_n^*, \dots, \epsilon_3^*$ of

X^\perp may be chosen so that ϵ_i is orthogonal to all the ϵ_j and $\omega(\epsilon_i, \epsilon_j^*) = \delta_{ij}$, where δ_{ij} is 1 if $i = j$ and 0 otherwise.

We consider the ordered basis

$$\epsilon_1, \epsilon_2, \dots, \epsilon_n; \epsilon_n^*, \dots, \epsilon_2^*, \epsilon_1^*,$$

of \mathbb{Q}^n , and define the standard Borel subgroup of $\mathrm{Sp}_n(\mathbb{Z})$ as the group of upper triangular matrices in Sp_n , and a maximal torus to be the group of diagonals in Sp_n . This datum determines a positive system of roots Φ^+ in the character group of T . We can then talk of highest and second highest roots in Φ^+ .

Consider the unipotent upper triangular matrices U_n in Sp_n with respect to the ordered basis of the preceding paragraph. The subgroup of U_n which acts trivially on ϵ_j, ϵ_j^* if $j \neq 1, 2$, is the group generated by the highest and a (actually there is only one) second highest root groups in U_n .

It is then immediate from Corollary 2.4 that the group generated by C_1, C_2, C_3 intersects the highest and a second highest root groups of $\mathrm{Sp}_n(\mathbb{Z})$ non-trivially. By assumption, Γ is a Zariski dense subgroup containing C_1, C_2, C_3 . By Theorem (3.5) of [Ve], it follows that Γ has finite index in $\mathrm{Sp}_n(\mathbb{Z})$.

3. PROOF OF THEOREM 1.1

3.1. Some Generalities. Consider the group Γ generated by the companion matrices A, B . Put $C = A^{-1}B$. Then, $(C - 1)(\mathbb{Z}^n) = \mathbb{Z}v$ for some vector v . Since C is identity on e_1, \dots, e_{n-1} this means that $(C - 1)e_n = \lambda v$ is a non-zero multiple of v .

Recall by [BH], that Γ preserves a symplectic nondegenerate form Ω on \mathbb{Z}^n . Write, for $x, y \in \mathbb{Z}^n$, $x.y$ for the number $\Omega(x, y)$.

Lemma 3.1. *The vector v is orthogonal to all of e_1, e_2, \dots, e_{n-1} . Moreover, v is a cyclic vector for the action of A on \mathbb{Q}^n .*

Proof. If $i \leq n-1$, then $e_i.e_n = Ce_i.Ce_n = e_i.(e_n + \lambda e_n) = e_i.e_n + \lambda e_i.v$. Cancelling $e_i.e_n$ on both the extreme sides of these equations, we get $v.e_i = 0$ for $i \leq n-1$.

If v is not cyclic, then there exists a polynomial ϕ in X of degree strictly less than n such that $\phi(A)v = 0$. If we view $\mathbb{Q}^n = \mathbb{Q}[X]/< f(X) >$, the action of A as multiplication by X , and $v = \frac{h}{X}$. Therefore, $\phi(A)v = 0$ implies that $\phi(X)\frac{h}{X}$ is divisible by $f(X)$. Since f, g are

coprime, then so are h, f and hence the divisibility of $\phi \frac{h}{X}$ by f implies that f divides ϕ , contradicting the assumption that ϕ has degree less than n .

□

Since A and B are companion matrices with minimal polynomials f, g , it follows after an easy computation, that

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & A_1 - B_1 \\ 0 & 1 & 0 & \cdots & A_2 - B_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & A_{n-1} - B_{n-1} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix},$$

where 1_{n-1} is the identity $(n-1) \times (n-1)$ -matrix, and v (resp. 0) is viewed as a column (resp. row) vector of length $n-1$.

Let

$$h = f - g = cX^{n-k} + c_{n-k-1}X^{n-k-1} + \cdots + c_{k+1}X^{k+1} + c_kX^k.$$

Since f, g are reciprocal polynomials, so is h and $c_k = c$ and $2k \leq n$. The vector v is a linear combination of e_1, \dots, e_{n-1} :

$$v = \sum_{j=k}^{n-k} c_j e_j.$$

Then,

$$A^k(v) = ce_n + \sum_{j=2k}^{n-k-1} c_{j-k} e_j.$$

Moreover, it follows from Lemma 3.1, that

$$\Omega(v, A^{-k}v) = \Omega(A^k v, v) = c\Omega(e_n, v) \neq 0.$$

Lemma 3.2. *Consider the two sets of three vectors $S_A = \{v, A^k v, A^{-k} v\}$ and $S_B = \{v, B^k v, B^{-k} v\}$. Then, the vectors in either S_A or in S_B are linearly independent.*

Proof. The vectors $v, A^k v$ are linearly independent, since any linear dependence implies that a polynomial in A of degree k annihilates v , contradicting Lemma 3.1.

Suppose that $v, A^k v, A^{-k} v$ are linearly dependent. Then a polynomial of the form $p(X^k)$ with p of degree two is such that $p(A^k)v = 0$. By Lemma 3.1, v is cyclic and $2k \leq n$ and hence $f(X) = p(X^k)$ with

$$k = \frac{n}{2}.$$

Similarly, we have a polynomial q of degree two such that $g(X) = q(X^k)$ if $v, B^k v, B^{-k} v$ are linearly dependent.

The conclusion of the last two paragraphs contradicts the assumption that f, g form a primitive pair. Therefore, the lemma follows. \square

We assume henceforth, that $w_1 = v, w_2 = A^{-k}v, w_3 = A^k v$ are linearly independent. Let W denote the \mathbb{Q} -vector space spanned by the $\{w_i\}$ and let $W_{\mathbb{Z}}$ denote the integral linear span of the w_i . Consider the transvections $C_1 = C = A^{-1}B$, $C_2 = A^{-k}CA^k$ and $C_3 = A^kCA^{-k}$. The images of $C_i - 1$ are spanned by w_1, w_2, w_3 .

Lemma 3.3. *The \mathbb{Z} span of w_1, w_2 is stable under the action of C_1, C_2 . Moreover, with respect to this basis, C_1, C_2 have (respectively) the matrix form*

$$M_1 = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

If $|c| \leq 2$, then as linear transformations on the \mathbb{Q} -span of w_1, w_2 , the group generated by M_1, M_2 is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Recall that $A^{-k}v = -ce_n + v'$ where v' is a linear combination of e_1, \dots, e_{n-1} . Hence C fixes v' and takes $-ce_n$ to $-c(e_n + v) = -ce_n - cv$. Therefore, $CA^{-k}v = A^{-k}v - cv$, i.e., $Cw_2 = w_2 - cw_1$. Moreover, $Cv = v$, i.e., $Cw_1 = w_1$.

The matrix form of $A^{-k}CA^k$ is similarly determined.

If $|c| \leq 2$, it is well known that the group generated by the two matrices M_1 and M_2 generate a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. \square

3.2. Proof of Theorem 1.1. In the preceding subsection, we have already verified that the vectors w_1, w_2, w_3 are linearly independent, and that $\Omega(w_1, w_2) \neq 0$. We will now verify that the group generated by the transvections C_1, C_2, C_3 satisfies the condition of Theorem 1.2.

We need only check that the group H generated by the restrictions C'_i to W of the C_i , contains a non-trivial element of the unipotent radical of Sp_W .

Let $W \cap W^\perp = \mathbb{Q}e$ for some $e \in W \setminus \{0\}$. Denote by w'_1, w'_2 the images of w_1, w_2 under the quotient map $W \rightarrow W/\mathbb{Q}e$. Since $\Omega(w_1, w_2) \neq 0$, it

follows that the span of w_1, w_2 intersects trivially with $\mathbb{Q}e$. By Lemma 3.3, the group generated by C_1'', C_2'' in $\text{GL}(W/\mathbb{Q}e)$ is an arithmetic group D . In particular, if $u \in \text{GL}(W/\mathbb{Q}e)$ is a unipotent element, then some power of u lies in the arithmetic group D . Therefore, some power $(C_3'')^m$ of the unipotent element C_3'' lies in D .

With respect to the basis e, w_1, w_2 , the restriction C_3' of C_3 to W has the matrix form

$$C_3' = \begin{pmatrix} 1 & w \\ 0_2 & C_3'' \end{pmatrix} = \begin{pmatrix} 1 & w \\ 0_2 & g \end{pmatrix},$$

where $w \in \mathbb{Q}^2$ may be viewed as an element of the unipotent radical of Sp_W and 0_2 is the zero column vector of length 2. A computation shows that

$$(C_3')^m = \begin{pmatrix} 1 & w_m \\ 0_2 & g^m \end{pmatrix},$$

where $w_m = w(1 + g + \cdots + g^{m-1}) \in \mathbb{Q}^2$ with $g = C_3''$ (w is a 1×2 matrix and we multiply it on the right by the 2×2 -matrix $1 + g + \cdots + g^{m-1}$ to get w_m).

Since $\Omega(w_1, w_3) = \Omega(v, A^k v) = \Omega(A^{-k} v, v) \neq 0$, it follows that the image $(C_3 - 1)w_1$ is a non-zero multiple of w_3 ; since w_3 is not a linear combination of w_1, w_2 (Lemma 3.2), it follows that $C_3'(w_1) = x_0 e + x_1 w_1 + x_2 w_2$, with $x_0 \neq 0$. Therefore, the vector $w \in \mathbb{Q}^2$ is non-zero.

Since C_3' is unipotent, so is $g = C_3''$; hence $1 + g + \cdots + g^{m-1} = \prod (g - \omega)$ (where the product over all ω which are non-trivial m -th roots of unity), is non-singular and hence $w(1 + g + \cdots + g^{m-1}) \neq 0$.

The group generated by C_1'' and C_2'' contains g^m . Hence the group generated by C_1' and C_2' contains an element of the form

$$h = \begin{pmatrix} 1 & 0 \\ 0_2 & g^m \end{pmatrix},$$

where 0 in the first row is the zero row vector in \mathbb{Q}^2 .

Therefore, multiplying $(C_3')^m$ on the right by the element h^{-1} we get that $(C_3')^m h^{-1}$ has the matrix form

$$(C_3')^m h^{-1} = \begin{pmatrix} 1 & w_m g^{-m} \\ 0_2 & 1 \end{pmatrix}.$$

This clearly lies in the unipotent radical of Sp_W . It is non-trivial: since w_m is non-zero and g^{-m} is non-singular, the row vector $w_m g^{-m}$ is non-zero.

Therefore all the conditions of Theorem 1.2 are satisfied, and hence, by Theorem 1.2, the hypergeometric group Γ is arithmetic. This proves Theorem 1.1.

4. REMARKS

[1] Computations show that when $n = 4$, For a large number of cases, the polynomial $h = f - g$ has leading coefficient ≤ 2 (assuming A, B are quasi-unipotent). In these cases, by Theorem 1.1, the monodromy group Γ is arithmetic.

In several other cases, $\frac{f-g}{X} = 3X^2 - X + 3$. Although, in this case the leading coefficient is 3, the monodromy group Γ is still arithmetic (see Example 48 of Table 4.2; it is shown by replacing the condition of Theorem 1.1, with the condition that for some $g \in \Gamma$, the coefficient of e_n in gv is ± 2 or ± 1). The proof proceeds in exactly the same way, by replacing $v, A^k v, A^{-k} v$ by the vectors $v, gv, g^{-1}v$.

[2] In Beukers-Heckman ([BH]) the connection with hypergeometric equation is explained. We recall it briefly.

Set $\theta = z \frac{d}{dz}$, viewed as a differential operator on the space $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let α be the n -tuple $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and β the complex n -tuple $(\beta_1, \dots, \beta_n)$. The differential operator

$$D = (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n),$$

is again a differential operator on $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The fundamental group of S (with base point, say, $\frac{1}{2}$), acts on the space of solutions of this differential equation. The fundamental group of S is generated by h_∞, h_0, h_1 where h_∞ is represented by a small loop going around ∞ exactly once. Define similarly h_0 and h_1 . Then $h_0 h_1 h_\infty = 1$.

Let, for $j \leq n$, $a_j = e^{2\pi i \alpha_j}$, $\beta_j = e^{2\pi i \beta_j}$. Take $f = \prod_{j=1}^n (X - a_j)$ and $g = \prod_{j=1}^n (X - b_j)$. We can then form the associated companion matrices A, B as in the introduction.

It is a result of Levelt (see [BH]), that there exist a basis $\{u\}$ of solutions of the differential equation $Du = 0$ on the curve S , with respect to which the monodromy action by the fundamental group of S

is described by the following. The action of h_∞ is by B^{-1} , that of h_0 is by A ; then h_1 acts by $C = A^{-1}B$.

The matrices A and B are quasi-unipotent exactly when α_j and β_j are rational numbers.

In the following subsection we give tables for the polynomials f, g when $n = 4$, (f, g) form a primitive pair, $f(0) = g(0) = 1$ and A, B are quasi-unipotent matrices with integral entries. We will also specify the case when we are able to prove that the associated monodromy group is arithmetic, using Theorem 1.1.

TABLE 4.1. Complete list of primitive *Symplectic* pairs of polynomials of degree 4 (and product of cyclotomic polynomials), for which the arithmeticity follows from our theorem

No.	$f(X)$	$g(X)$	α	β	$f(X) - g(X)$
1	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$0,0,0,0$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$-2X^3 + 3X^2 - 2X$
2	$X^4 - 2X^2 + 1$	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$-2X^3 - 5X^2 - 2X$
3	$X^4 - 2X^2 + 1$	$X^4 + X^3 + 2X^2 + X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}$	$-X^3 - 4X^2 - X$
4	$X^4 - 2X^2 + 1$	$X^4 + X^3 + X^2 + X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$-X^3 - 3X^2 - X$
5	$X^4 - 2X^2 + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$2X^3 - 5X^2 + 2X$
6	$X^4 - 2X^2 + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$X^3 - 4X^2 + X$
7	$X^4 - 2X^2 + 1$	$X^4 - X^3 + X^2 - X + 1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$X^3 - 3X^2 + X$
8	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$2X^3 + 3X^2 + 2X$
9	$X^4 - X^3 - X + 1$	$X^4 + 2X^2 + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$-X^3 - 2X^2 - X$
10	$X^4 - X^3 - X + 1$	$X^4 + X^3 + X^2 + X + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$-2X^3 - X^2 - 2X$
11	$X^4 - X^3 - X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$X^3 - 3X^2 + X$
12	$X^4 - X^3 - X + 1$	$X^4 + X^3 + X + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$-2X^3 - 2X$
13	$X^4 - X^3 - X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$-2X^2$
14	$X^4 - X^3 - X + 1$	$X^4 + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-X^3 - X$
15	$X^4 - X^3 - X + 1$	$X^4 - X^3 + X^2 - X + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$-X^2$
16	$X^4 - X^3 - X + 1$	$X^4 - X^2 + 1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-X^3 + X^2 - X$
17	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 + 2X^2 + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$2X^3 + X^2 + 2X$
18	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	X^2
19	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$X^3 + 2X^2 + X$
20	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 + X^3 + X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$X^3 + 3X^2 + X$
21	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$2X^3 + 3X^2 + 2X$
22	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - X^2 + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2X^3 + 4X^2 + 2X$
23	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$2X^3 + 3X^2 + 2X$
24	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$-2X^2$
25	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 + X^2 + 1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}$	$-2X^3 + X^2 - 2X$
26	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 + 1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-2X^3 + 2X^2 - 2X$
27	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 - X^3 + X^2 - X + 1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$-X^3 + X^2 - X$
28	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 - X^2 + 1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-2X^3 + 3X^2 - 2X$
29	$X^4 + 2X^2 + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$-X^3 + X^2 - X$
30	$X^4 + 2X^2 + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$2X^3 - X^2 + 2X$

Table 4.1 continued...

Table 4.1 continued...

No.	$f(X)$	$g(X)$	α	β	$f(X) - g(X)$
31	$X^4 + 2X^2 + 1$	$X^4 + X^3 + X + 1$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	$-X^3 + 2X^2 - X$
32	$X^4 + 2X^2 + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$X^3 + X^2 + X$
33	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$X^3 + X^2 + X$
34	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 + X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}, \frac{5}{6}$	$2X^3 + X^2 + 2X$
35	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$2X^3 + 2X^2 + 2X$
36	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 - X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2X^3 + 3X^2 + 2X$
37	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{4}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	X^2
38	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 + X^3 + X + 1$	$\frac{1}{4}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	$2X^2$
39	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 + 1$	$\frac{1}{4}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$X^3 + 2X^2 + X$
40	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{4}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2X^3 + X^2 + 2X$
41	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 - X^2 + 1$	$\frac{1}{4}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$X^3 + 3X^2 + X$
42	$X^4 + X^3 + X^2 + X + 1$	$X^4 + X^3 + X + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	X^2
43	$X^4 + X^3 + X^2 + X + 1$	$X^4 + X^2 + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{3}, \frac{2}{3}, \frac{5}{6}, \frac{5}{6}$	$X^3 + X$
44	$X^4 + X^3 + X^2 + X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{4}, \frac{3}{4}, \frac{5}{6}, \frac{5}{6}$	$2X^3 - X^2 + 2X$
45	$X^4 + X^3 + X^2 + X + 1$	$X^4 + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$X^3 + X^2 + X$
46	$X^4 + X^3 + X^2 + X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2X^3 + 2X$
47	$X^4 + X^3 + X^2 + X + 1$	$X^4 - X^2 + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$X^3 + 2X^2 + X$
48	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$X^4 - X^3 + X^2 - X + 1$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$-2X^3 + 3X^2 - 2X$
49	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$X^4 + 1$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-2X^3 + 3X^2 - 2X$
50	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$-X^3 + 2X^2 - X$
51	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$X^4 - X^2 + 1$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-22X^3 + 4X^2 - 2X$
52	$X^4 + X^3 + X + 1$	$X^4 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-X^3 - X$
53	$X^4 + X^3 + X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2X^3 - X^2 + 2X$
54	$X^4 + X^3 + X + 1$	$X^4 - X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$X^3 + X^2 + X$
55	$X^4 + X^2 + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$X^3 + X$
56	$X^4 - X^3 + 2X^2 - X + 1$	$X^4 + 1$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-X^3 + 2X^2 - X$
57	$X^4 - X^3 + 2X^2 - X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	X^2
58	$X^4 - X^3 + 2X^2 - X + 1$	$X^4 - X^2 + 1$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-X^3 + 3X^2 - X$
59	$X^4 + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	X^2
60	$X^4 - X^3 + X^2 - X + 1$	$X^4 - X^2 + 1$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-X^3 + 2X^2 - X$

TABLE 4.2. Complete list of primitive *Symplectic* pairs of polynomials of degree 4 (and product of cyclotomic polynomials), for which we don't know how to prove the arithmeticity

No.	$f(X)$	$g(X)$	α	β	$f(X) - g(X)$
1	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 4X^3 + 6X^2 + 4X + 1$	0,0,0,0	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$-8X^3 - 8X$
2	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 2X^3 + 3X^2 + 2X + 1$	0,0,0,0	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$-6X^3 + 3X^2 - 6X$
3	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 3X^3 + 4X^2 + 3X + 1$	0,0,0,0	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$-7X^3 + 2X^2 - 7X$
4	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 2X^2 + 1$	0,0,0,0	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$-4X^3 + 4X^2 - 4X$
5	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 2X^3 + 2X^2 + 2X + 1$	0,0,0,0	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$-6X^3 + 4X^2 - 6X$
6	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + X^3 + 2X^2 + X + 1$	0,0,0,0	$\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$-5X^3 + 4X^2 - 5X$
7	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + X^3 + X^2 + X + 1$	0,0,0,0	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$-5X^3 + 5X^2 - 5X$
8	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + -2X^3 + 3X^2 - 2X + 1$	0,0,0,0	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$-2X^3 + 3X^2 - 2X$
9	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + X^3 + X + 1$	0,0,0,0	$\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$-5X^3 + 6X^2 - 5X$
10	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + X^2 + 1$	0,0,0,0	$\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}$	$-4X^3 + 5X^2 - 4X$
11	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	0,0,0,0	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$-3X^3 + 4X^2 - 3X$
12	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 + 1$	0,0,0,0	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-4X^3 + 6X^2 - 4X$
13	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 - X^3 + X^2 - X + 1$	0,0,0,0	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$-3X^3 + 5X^2 - 3X$
14	$X^4 - 4X^3 + 6X^2 - 4X + 1$	$X^4 - X^2 + 1$	0,0,0,0	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-4X^3 + 7X^2 - 4X$
15	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - X^3 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 0, \frac{1}{3}, \frac{2}{3}$	$5X^3 + 6X^2 + 5X$
16	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 0, \frac{1}{4}, \frac{3}{4}$	$6X^3 + 4X^2 + 6X$
17	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + 2X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$4X^3 + 4X^2 + 4X$
18	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + X^3 + 2X^2 + X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}$	$3X^3 + 4X^2 + 3X$
19	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + X^3 + X^2 + X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$3X^3 + 5X^2 + 3X$
20	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$7X^3 + 2X^2 + 7X$
21	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$6X^3 + 3X^2 + 6X$
22	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}$	$4X^3 + 5X^2 + 4X$
23	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$5X^3 + 4X^2 + 5X$
24	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$4X^3 + 6X^2 + 4X$
25	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$5X^3 + 5X^2 + 5X$
26	$X^4 + 4X^3 + 6X^2 + 4X + 1$	$X^4 - X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$4X^3 + 7X^2 + 4X$
27	$X^4 - X^3 - X + 1$	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$0, 0, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$-3X^3 - 2X^2 - 3X$
28	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$0, 0, \frac{1}{4}, \frac{3}{4}$	$4X^3 + X^2 + 4X$
29	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$5X^3 - X^2 + 5X$
30	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$4X^3 + 4X$

Table 4.2 continued...

Table 4.2 continued...

No.	$f(X)$	$g(X)$	α	β	$f(X) - g(X)$
31	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$3X^3 + X^2 + 3X$
32	$X^4 + 2X^3 + 3X^2 + 2X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$3X^3 + 2X^2 + 3X$
33	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$0, 0, \frac{1}{4}, \frac{3}{4}$	$5X^3 + 2X^2 + 5X$
34	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 + 2X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$3X^3 + 2X^2 + 3X$
35	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$6X^3 + 6X$
36	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$5X^3 + X^2 + 5X$
37	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - X^3 + 2X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}$	$4X^3 + 2X^2 + 4X$
38	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$3X^3 + 4X^2 + 3X$
39	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$4X^3 + 3X^2 + 4X$
40	$X^4 + 3X^3 + 4X^2 + 3X + 1$	$X^4 - X^2 + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$3X^3 + 5X^2 + 3X$
41	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 + X^3 + X^2 + X + 1$	$0, 0, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$-3X^3 + X^2 - 3X$
42	$X^4 - 2X^3 + 2X^2 - 2X + 1$	$X^4 + X^3 + X + 1$	$0, 0, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}$	$-3X^3 + 2X^2 - 3X$
43	$X^4 + 2X^2 + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$3X^3 - 2X^2 + 3X$
44	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$5X^3 - 2X^2 + 5X$
45	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$4X^3 - X^2 + 4X$
46	$X^4 + 2X^3 + 2X^2 + 2X + 1$	$X^4 - X^3 + X^2 - X + 1$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$3X^3 + X^2 + 3X$
47	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$4X^3 - 2X^2 + 4X$
48	$X^4 + X^3 + 2X^2 + X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$3X^3 - X^2 + 3X$
49	$X^4 + X^3 + X^2 + X + 1$	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$4X^3 - 3X^2 + 4X$
50	$X^4 + X^3 + X^2 + X + 1$	$X^4 - 2X^3 + 3X^2 - 2X + 1$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$3X^3 - 2X^2 + 3X$
51	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$X^4 + 1$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$-3X^3 + 4X^2 - 3X$
52	$X^4 - 3X^3 + 4X^2 - 3X + 1$	$X^4 - X^2 + 1$	$0, 0, \frac{1}{6}, \frac{5}{6}$	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$-3X^3 + 5X^2 - 3X$

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